

$$C_l^{-1} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{31}} \end{bmatrix} \quad (19.2.4)$$

where the subscripts denote the material axes, i.e.,

$$\nu_{ij} = \nu_{x'_i x'_j} \quad \text{and} \quad E_{ii} = E_{x'_i} \quad (19.2.5)$$

Since C_l is symmetric

$$\frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}}, \text{ etc.} \quad (19.2.6)$$

The vector of Green-St. Venant strain components is

$$E^t = [E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, E_{31},] \quad (19.2.7)$$

After computing S_{ij} , we use Equation (18.32) to obtain the Cauchy stress. This model will predict realistic behavior for finite displacement and rotations as long as the strains are small.

Material Model 3: Elastic Plastic with Kinematic Hardening

Isotropic, kinematic, or a combination of isotropic and kinematic hardening may be obtained by varying a parameter, called β between 0 and 1. For β equal to 0 and 1, respectively, kinematic and isotropic hardening are obtained as shown in Figure 19.3.1. Krieg and Key [1976] formulated this model and the implementation is based on their paper.

In isotropic hardening, the center of the yield surface is fixed but the radius is a function of the plastic strain. In kinematic hardening, the radius of the yield surface is fixed but the center translates in the direction of the plastic strain. Thus the yield condition is

$$\phi = \frac{1}{2} \xi_{ij} \xi_{ij} - \frac{\sigma_y^2}{3} = 0 \quad (19.3.1)$$

where

$$\xi_{ij} = s_{ij} - \alpha_{ij} \quad (19.3.2)$$

$$\sigma_y = \sigma_0 + \beta E_p \varepsilon_{eff}^p \quad (19.3.3)$$

The co-rotational rate of α_{ij} is

$$\alpha_{ij}^{\nabla} = (1-\beta) \frac{2}{3} E_p \dot{\varepsilon}_{ij}^p \quad (19.3.4)$$

Hence,

$$\alpha_{ij}^{n+1} = \alpha_{ij}^n + \left(\alpha_{ij}^{\nabla n+1/2} + \alpha_{ik}^n \Omega_{kj}^{n+1/2} + \alpha_{jk}^n \Omega_{ki}^{n+1/2} \right) \Delta t^{n+1/2}. \quad (19.3.5)$$

Strain rate is accounted for using the Cowper-Symonds [Jones 1983] model which scales the yield stress by a strain rate dependent factor

$$\sigma_y = \left[1 + \left(\frac{\dot{\varepsilon}}{C} \right)^{\frac{1}{p}} \right] (\sigma_0 + \beta E_p \varepsilon_{eff}^p) \quad (19.3.6)$$

where p and C are user defined input constants and $\dot{\varepsilon}$ is the strain rate defined as:

$$\dot{\varepsilon} = \sqrt{\dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}} \quad (19.3.7)$$

The current radius of the yield surface, σ_y , is the sum of the initial yield strength, σ_0 , plus the growth $\beta E_p \varepsilon_{eff}^p$, where E_p is the plastic hardening modulus

$$E_p = \frac{E_t E}{E - E_t} \quad (19.3.8)$$

and ε_{eff}^p is the effective plastic strain

$$\varepsilon_{eff}^p = \int_0^t \left(\frac{2}{3} \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p \right)^{1/2} dt \quad (19.3.9)$$

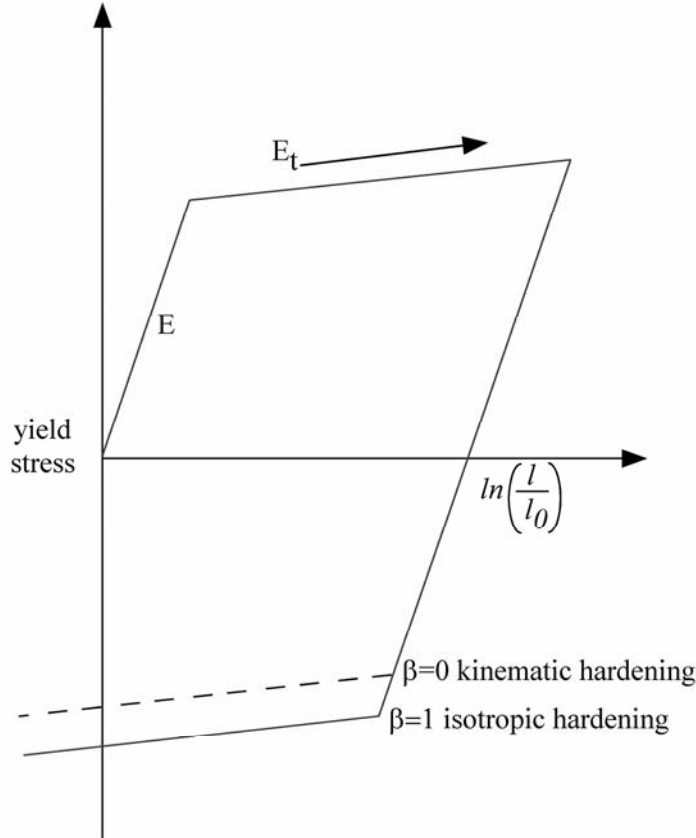


Figure 19.3.1. Elastic-plastic behavior with isotropic and kinematic hardening where l_0 and l are the undeformed and deformed length of uniaxial tension specimen, respectively.

The plastic strain rate is the difference between the total and elastic (right superscript e) strain rates:

$$\dot{\epsilon}_{ij}^p = \dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^e \tag{19.3.10}$$

In the implementation of this material model, the deviatoric stresses are updated elastically, as described for model 1, but repeated here for the sake of clarity:

$$\sigma_{ij}^* = \sigma_{ij}^n + C_{ijkl} \Delta \epsilon_{kl} \tag{19.3.11}$$

where

- σ_{ij}^* is the trial stress tensor,
- σ_{ij}^n is the stress tensor from the previous time step,
- C_{ijkl} is the elastic tangent modulus matrix,
- $\Delta \epsilon_{kl}$ is the incremental strain tensor.

and, if the yield function is satisfied, nothing else is done. If, however, the yield function is violated, an increment in plastic strain is computed, the stresses are scaled back to the yield surface, and the yield surface center is updated.

Let s_{ij}^* represent the trial elastic deviatoric stress state at n+1

$$s_{ij}^* = \sigma_{ij}^* - \frac{1}{3} \sigma_{kk}^* \quad (19.3.12)$$

and

$$\xi_{ij}^* = s_{ij}^* - \alpha_{ij} \quad (19.3.13)$$

Define the yield function,

$$\phi = \frac{3}{2} \xi_{ij}^* \xi_{ij}^* - \sigma_y^2 = \Lambda^2 - \sigma_y^2 \begin{cases} \leq 0 & \text{for elastic or neutral loading} \\ > 0 & \text{for plastic hardening} \end{cases} \quad (19.3.14)$$

For plastic hardening then

$$\varepsilon_{eff}^{p^{n+1}} = \varepsilon_{eff}^{p^n} + \frac{\Lambda - \sigma_y}{3G + E_p} = \varepsilon_{eff}^{p^n} + \Delta\varepsilon_{eff}^p \quad (19.3.15)$$

scale back the stress deviators

$$\sigma_{ij}^{n+1} = \sigma_{ij}^* - \frac{3G\Delta\varepsilon_{eff}^p}{\Lambda} \xi_{ij}^* \quad (19.3.16)$$

and update the center:

$$\alpha_{ij}^{n+1} = \alpha_{ij}^n + \frac{(1-\beta)E_p}{\Lambda} \Delta\varepsilon_{eff}^p \xi_{ij}^* \quad (19.3.17)$$

Plane Stress Plasticity

The plane stress plasticity options apply to beams, shells, and thick shells. Since the stresses and strain increments are transformed to the lamina coordinate system for the constitutive evaluation, the stress and strain tensors are in the local coordinate system.

The application of the Jaumann rate to update the stress tensor allows for the possibility that the normal stress, σ_{33} , will not be zero. The first step in updating the stress tensor is to compute a trial plane stress update assuming that the incremental strains are elastic. In the above, the normal strain increment $\Delta\varepsilon_{33}$ is replaced by the elastic strain increment

$$\Delta\varepsilon_{33} = -\frac{\sigma_{33} + \lambda(\Delta\varepsilon_{11} + \Delta\varepsilon_{22})}{\lambda + 2\mu} \quad (19.3.18)$$

where λ and μ are Lamé's constants.

When the trial stress is within the yield surface, the strain increment is elastic and the stress update is completed. Otherwise, for the plastic plane stress case, secant iteration is used to solve Equation (19.3.16) for the normal strain increment ($\Delta\epsilon_{33}$) required to produce a zero normal stress:

$$\sigma_{33}^i = \sigma_{33}^* - \frac{3G\Delta\epsilon_{eff}^{p^i} \xi_{33}}{\Lambda} \quad (19.3.19)$$

Here, the superscript i indicates the iteration number.

The secant iteration formula for $\Delta\epsilon_{33}$ (the superscript p is dropped for clarity) is

$$\Delta\epsilon_{33}^{i+1} = \Delta\epsilon_{33}^{i-1} - \frac{\Delta\epsilon_{33}^i - \Delta\epsilon_{33}^{i-1}}{\sigma_{33}^i - \sigma_{33}^{i-1}} \sigma_{33}^{i-1} \quad (19.3.20)$$

where the two starting values are obtained from the initial elastic estimate and by assuming a purely plastic increment, i.e.,

$$\Delta\epsilon_{33}^1 = -(\Delta\epsilon_{11} - \Delta\epsilon_{22}) \quad (19.3.21)$$

These starting values should bound the actual values of the normal strain increment.

The iteration procedure uses the updated normal strain increment to update first the deviatoric stress and then the other quantities needed to compute the next estimate of the normal stress in Equation (19.3.19). The iterations proceed until the normal stress σ_{33}^i is sufficiently small. The convergence criterion requires convergence of the normal strains:

$$\frac{|\Delta\epsilon_{33}^i - \Delta\epsilon_{33}^{i-1}|}{|\Delta\epsilon_{33}^{i+1}|} < 10^{-4} \quad (19.3.22)$$

After convergence, the stress update is completed using the relationships given in Equations (19.3.16) and (19.3.17)

Material Model 4: Thermo-Elastic-Plastic

This model was adapted from the NIKE2D [Hallquist 1979] code. A more complete description of its formulation is given in the NIKE2D user's manual.

Letting T represent the temperature, we compute the elastic co-rotational stress rate as

$$\sigma_{ij}^{\nabla} = C_{ijkl} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^T) + \dot{\theta}_{ij} dT \quad (19.4.1)$$

where